MAPMAKING WITH REAL HALF-WAVE PLATES

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Here we derive the complete equations for constructing maps from a telescope with a non-ideal half-wave plate to modulate polarization. This text follows the equations and conventions in [1, 2].

1 GENERAL FORMALISM

We define the radiation state along a direction of propagation $\hat{\mathbf{r}}$ as a Stokes vector $\mathbf{s}(\hat{\mathbf{r}})$, where

$$\mathbf{s}^\mathsf{T}(\hat{\mathbf{r}}) = \begin{pmatrix} I(\hat{\mathbf{r}}) & Q(\hat{\mathbf{r}}) & U(\hat{\mathbf{r}}) & V(\hat{\mathbf{r}}) \end{pmatrix}. \tag{1}$$

A polarization-sensitive bolometer can be treated in the Mueller matrix formalism as a polarization-sensitive element with Muller matrix $\mathbf{M}$, followed by an incoherent absorber $\mathbf{a} = (1, 0, 0, 0)$ that is sensitive only to Stokes intensity. The signal d on the bolometer due to incident radiation $\mathbf{s}$ along the direction $\hat{\mathbf{r}}$ is thus, up to an overall gain factor,

$$d = \mathbf{a}^\mathsf{T}\mathbf{M}\,\mathbf{s} = M_{II}\,I + M_{IQ}\,Q + M_{IU}\,U + M_{IV}\,V. \tag{2}$$

We can write the model of eq. (2) in matrix form (ignoring additive noise for clarity) for a sequence of N_{samp} measurements as

$$\mathbf{d} = \mathbf{A}\mathbf{m}. \tag{3}$$

In this case, $\mathbf{d}$ is a vector of length N_{samp}, where each element is a single data sample. Similarly, the underlying Stokes (I, Q, U, V) parameters are interleaved into a single discretely pixelized map vector $\mathbf{m}$ of length $4N_{pix}$, with $N_{pix} \ll N_{samp}$. Explicitly, we have

$$\begin{aligned} \mathbf{m}^\mathsf{T} &= \begin{pmatrix} \mathbf{s}_0^\mathsf{T} & \cdots & \mathbf{s}_{N_{pix}}^\mathsf{T} \end{pmatrix} \\ &= \begin{pmatrix} I_0 & Q_0 & U_0 & V_0 & \cdots & I_{N_{pix}} & Q_{N_{pix}} & U_{N_{pix}} & V_{N_{pix}} \end{pmatrix}. \end{aligned} \tag{4}$$

Finally, $\mathbf{A}$ is a sparse matrix with $N_{samp} \times 4N_{pix}$ elements that maps each set of four Stokes parameters to a data sample according to eq. (2). We often compute the quantities

$\boldsymbol{v} = \mathbf{A}^{\mathsf{T}}\mathbf{d}$ and $\mathbf{P} = \mathbf{A}^{\mathsf{T}}\mathbf{A}$ for each map pixel in the process of solving eq. (3). More explicitly, these quantities are:

$$\boldsymbol{v}_p = \sum_{\hat{\mathbf{r}} \in p} \begin{pmatrix} M_{II}\, d \\ M_{IQ}\, d \\ M_{IU}\, d \\ M_{IV}\, d \end{pmatrix} \tag{5}$$

and

$$\mathbf{P}_p = \sum_{\hat{\mathbf{r}} \in p} \begin{pmatrix} M_{II}^2 & M_{II}\, M_{IQ} & M_{II}\, M_{IU} & M_{II}\, M_{IV} \\ - & M_{IQ}^2 & M_{IQ}\, M_{IU} & M_{IQ}\, M_{IV} \\ - & - & M_{IU}^2 & M_{IU}\, M_{IV} \\ - & - & - & M_{IV}^2 \end{pmatrix}, \tag{6}$$

where each element is summing over all values that fall into the map pixel p.

In the following sections, we compute the Mueller matrix elements in several common mapmaking cases, with some recommendations for how to implement these in software efficiently. Sample indices and pointing directions are suppressed for clarity.

2 COUPLING OF A SINGLE POLARIZED INSTRUMENT

We begin with the matrix equation for a single polarized instrument aligned at an angle ψ relative to the sky coordinate axes. In the remainder of the text, this is what we call the "boresight" or instrument angle. The rotation of the instrument is encoded in the Mueller rotation matrix $\mathbf{M}_\psi$, defined as

$$\mathbf{M}_\psi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\psi) & \sin(2\psi) & 0 \\ 0 & -\sin(2\psi) & \cos(2\psi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7}$$

and we treat our instrument as a partial polarizer aligned with the vertical axis:

$$\mathbf{M}_{\mathrm{pol}} = \frac{1}{2}\begin{pmatrix} \eta^2+\delta^2 & \eta^2-\delta^2 & 0 & 0 \\ \eta^2-\delta^2 & \eta^2+\delta^2 & 0 & 0 \\ 0 & 0 & 2\eta\delta & 0 \\ 0 & 0 & 0 & 2\eta\delta \end{pmatrix}. \tag{8}$$

Without loss of generality, we can normalize this matrix by its M_{II} element, under the assumption that co-polar (η) and cross-polar (δ) quantities are constant, and that the overall gain of the instrument is calibrated by other means. We can now define the instrument's polarization efficiency as

$$\gamma = \frac{\eta^2-\delta^2}{\eta^2+\delta^2}, \tag{9}$$

so that the matrix of eq. (8) simplifies to:

$$\mathbf{M}_{\mathrm{pol}} = \begin{pmatrix} 1 & \gamma & 0 & 0 \\ \gamma & 1 & 0 & 0 \\ 0 & 0 & \sqrt{1-\gamma^2} & 0 \\ 0 & 0 & 0 & \sqrt{1-\gamma^2} \end{pmatrix}. \tag{10}$$

The complete Mueller matrix for this system is

$$\mathbf{M} = \mathbf{M}_{\mathrm{pol}}\,\mathbf{M}_{\psi}, \tag{11}$$

which results in the data model

$$d = I + \gamma\left(Q\cos(2\psi) + U\sin(2\psi)\right). \tag{12}$$

This is the simplest form for the signal observed by a polarization-sensitive absorber. If we also assume $\gamma \to 1$ in the ideal polarizer limit, we can now write down the elements of the ideal Mueller matrix $\widehat{\mathbf{M}}$, which depends solely on the instrument orientation ψ:

$$\widehat{M}_{II} = 1 \qquad \widehat{M}_{IQ} = \cos(2\psi) \qquad \widehat{M}_{IU} = \sin(2\psi) \qquad \widehat{M}_{IV} = 0. \tag{13}$$

3 COUPLING WITH MULTIPLE POLARIZERS

We now consider the coupling of multiple channels aligned at different angles ξ through a single telescope with a common boresight orientation ψ. The Mueller matrix for this system now includes an additional rotation

$$\mathbf{M} = \mathbf{M}_{\text{pol}}\, \mathbf{M}_{-\xi}\, \mathbf{M}_{\psi}. \tag{14}$$

Note that the sign of the channel angle used here differs from the convention in [1], based on empirical checks of TP correlations, assuming the angle sign conventions used by SPIDER. This Mueller matrix results in the data model

$$d = I + \gamma \left(Q \cos(2\psi - 2\xi) + U \sin(2\psi - 2\xi) \right). \tag{15}$$

Because the channel angle ξ is typically assumed to be a constant, we can separate the channel-dependent parts of the Mueller matrices, and write these in terms of the ideal matrix elements of eq. (13):

$$\begin{pmatrix} M_{II} \\ M_{IQ} \\ M_{IU} \\ M_{IV} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma\cos(2\xi) & \gamma\sin(2\xi) & 0 \\ 0 & -\gamma\sin(2\xi) & \gamma\cos(2\xi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \widehat{M}_{II} \\ \widehat{M}_{IQ} \\ \widehat{M}_{IU} \\ \widehat{M}_{IV} \end{pmatrix} \tag{16}$$

Note that the matrix eq. (16) is composed of two unique elements. This means that we can pre-compute these quantities once per channel, and apply them to the ideal quantities of eq. (13) that are typically computed for every sample of data as the instrument scans across the sky.

4 COUPLING THROUGH AN IDEAL HALF-WAVE PLATE

To modulate the polarization angle on the sky, we can add a half-wave plate (HWP) in front of the detector. The Mueller matrix for an ideal HWP aligned with the polarizer axis is

$$\mathbf{M}_{HWP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{17}$$

and that of an HWP rotated by an arbitrary angle θ is

$$\mathbf{M}_{HWP}(\theta) = \mathbf{M}_{-\theta}\,\mathbf{M}_{HWP}\mathbf{M}_{\theta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(4\theta) & \sin(4\theta) & 0 \\ 0 & \sin(4\theta) & -\cos(4\theta) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{18}$$

which results in the full Mueller matrix

$$\mathbf{M} = \mathbf{M}_{\text{pol}}\,\mathbf{M}_{-\xi}\,\mathbf{M}_{HWP}(\theta)\,\mathbf{M}_{\psi} \tag{19}$$

and data model

$$d = I + \gamma\left(Q\cos(2\psi + 4\theta + 2\xi) + U\sin(2\psi + 4\theta + 2\xi)\right). \tag{20}$$

Writing the Mueller matrix elements in terms of eq. (13) now gives

$$\begin{pmatrix} M_{II} \\ M_{IQ} \\ M_{IU} \\ M_{IV} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma\cos(4\theta + 2\xi) & -\gamma\sin(4\theta + 2\xi) & 0 \\ 0 & \gamma\sin(4\theta + 2\xi) & \gamma\cos(4\theta + 2\xi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \widehat{M}_{II} \\ \widehat{M}_{IQ} \\ \widehat{M}_{IU} \\ \widehat{M}_{IV} \end{pmatrix}. \tag{21}$$

For efficient mapmaking, these quantities can be computed and stored once per channel and per HWP angle.

5 COUPLING THROUGH A NON-IDEAL HALF-WAVE PLATE

In full generality, we can also introduce non-idealities in the HWP, in which case the HWP Mueller matrix becomes

$$\mathbf{M}_{HWP} = \begin{pmatrix} T & \rho & 0 & 0 \\ \rho & T & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{pmatrix}, \tag{22}$$

which reduces to the ideal case when $T = -c = 1$ and $\rho = s = 0$. We can again normalize by T to fold any DC effects into the overall channel calibration; to do so we define the reduced quantities $\overline{\rho} = \rho/T$, $\overline{c} = c/T$ and $\overline{s} = s/T$. The data model for this optical system is

$$M_{II} = 1 + \gamma\overline{\rho}\cos(2\theta + 2\xi), \tag{23}$$
$$M_{IQ} = \overline{\rho}\cos(2\theta + 2\psi) + \tfrac{1}{2}(1+\overline{c})\gamma\cos(2\psi - 2\xi) + \tfrac{1}{2}(1-\overline{c})\gamma\cos(2\psi + 4\theta + 2\xi), \tag{24}$$
$$M_{IU} = \overline{\rho}\sin(2\theta + 2\psi) + \tfrac{1}{2}(1+\overline{c})\gamma\sin(2\psi - 2\xi) + \tfrac{1}{2}(1-\overline{c})\gamma\sin(2\psi + 4\theta + 2\xi), \tag{25}$$
$$M_{IV} = \overline{s}\gamma\sin(2\theta + 2\xi). \tag{26}$$

These equations can again be written in terms of $\widehat{\mathbf{M}}$ as

$$\begin{pmatrix} M_{II} \\ M_{IQ} \\ M_{IU} \\ M_{IV} \end{pmatrix} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & -C & 0 \\ 0 & C & B & 0 \\ 0 & 0 & 0 & D \end{pmatrix} \begin{pmatrix} \widehat{M}_{II} \\ \widehat{M}_{IQ} \\ \widehat{M}_{IU} \\ \widehat{M}_{IV} \end{pmatrix}, \tag{27}$$

where the four unique functions are

$$A = 1 + \gamma\overline{\rho}\cos(2\theta + 2\xi), \tag{28}$$
$$B = \overline{\rho}\cos(2\theta) + \tfrac{1}{2}(1+\overline{c})\gamma\cos(2\xi) + \tfrac{1}{2}(1-\overline{c})\gamma\cos(4\theta + 2\xi), \tag{29}$$
$$C = \overline{\rho}\sin(2\theta) - \tfrac{1}{2}(1+\overline{c})\gamma\sin(2\xi) + \tfrac{1}{2}(1-\overline{c})\gamma\sin(4\theta + 2\xi), \tag{30}$$
$$D = \overline{s}\gamma\sin(2\theta + 2\xi). \tag{31}$$

With these equations in hand, we can either compute simulation timestreams using eq. (2) or construct the vector and matrix quantities eqs. (5) and (6) to solve for the underlying map.

6 IMPLEMENTATION IN UNIMAP

To implement these equations in `unimap`, we require some changes to both the `C` and `python` codes.

On the `C` end, we add an attribute called `mueller` of type `double[4]` to the `qp_det_t` structure, which will store the values of $A/B/C/D$ for each channel. This attribute is used in both the `qp_tod2map1` and `qp_map2tod1` functions as follows:

1. For each data sample, compute per-channel pixel number and ψ using `qp_quat2pix`, from a channel offset quaternion constructed from only the `az` and `el` pointing offsets. This produces `cpp` and `spp` parameters that are exactly the $\widehat{M}_{IQ}$ and $\widehat{M}_{IU}$ quantities of eq. (13).

2. Compute the Mueller matrix elements for the data sample using eq. (27), and then construct the `vec` and `proj` arrays using eqs. (5) and (6).

On the `python` end, we introduce four new bolotable columns for the $A/B/C/D$ parameters. These columns are updated in memory whenever the HWP angle is updated. Pre-computing these on disk isn't necessarily accurate, since there are cases where the actual HWP angle doesn't match the commanded value (on which the HWP index is based). We'd have to store bolotables with over 100 columns, which seems kind of unwieldy.

Note that this implementation includes the fourth Stokes V dimension for both data simulation and mapmaking.

BIBLIOGRAPHY

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[2] S. Bryan. *Half-wave Plates for the SPIDER Cosmic Microwave Background Polarimeter*. PhD thesis, Case Western Reserve University, February 2014. arXiv:1402.2591.